Random Stuff

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Outline

- Randomness around us

- How to make random numbers?

- Monte Carlo methods for:

  - **Simulation**
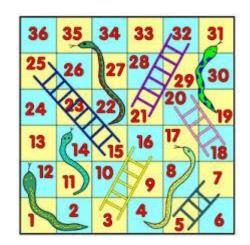
  - **Estimation**

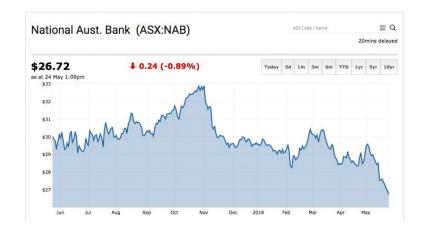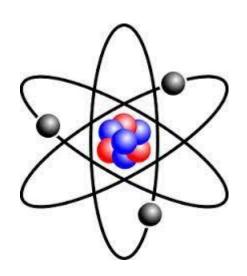 - **Optimisation**

Games of Chance

More Randomness

Randomness at its Deepest Level

We are such stuff as dreams are made on
And our little life is rounded with a sleep

William Shakespeare (The Tempest)

Monte Carlo (Monaco)

Monte Carlo Method

Stanislav Ulam   John von Neumann   Nicholas Metropolis

A MILLION
Random Digits

WITH

100,000 Normal Deviates

RAND

⭐☆☆☆☆ A Million Random Digits"? HA!

They only used 10, and just kept repeating them in different combinations!

Don't be fooled!

★★★★☆ Engrossing

A very engrossing book with historical importance, it keeps you guessing until the end.

★★★☆☆ Predictable

It seemed like about 10% of the time I was able to predict which number was next.

It was still better than Life of Pi which, aside from being irrational, included no estimations of Pi at all!

★★★★☆ Almost perfect

Such a terrific reference work! But with so many terrific random digits, it's a shame they didn't sort them, to make it easier to find the one you're looking for.

★★★★☆ Pretty good but needs some fixes for the next edition

This is a very thorough tome and I found myself unexpectedly enjoying the numbers. However the reason I'm giving the book 4 stars is that it's very difficult to find your way back to a number you had previously enjoyed.

The RAND Corporation and/or the publisher really need to add an index so you can quickly refer your way back to those highlights that so wonderfully brightened up the day.

⭐☆☆☆☆ **Shocked and appalled**

Those Deviates were in no way normal. This is a shocking display of what our country as a whole has come to. When I was a kid, we had standard deviates and we liked it.

★★☆☆☆ Normal deviates? I don't think so, pal.

Speaking as a normal deviate myself, I have to say this book is all wrong. I can't believe any real deviates were involved in researching it. What kind of weirdo has a fetish for numbers? More latex, less LaTeX, guys.

★★★☆☆   Wait for the movies …

I was thumbing through this in a book store…long enough to enjoy but certainly not long enough to *really* get the plot. Should I buy it or not?

Then a book store employee happened to mention that Peter Jackson was being approached to split the story into 3 parts and create a film trilogy based on the book. Enough said! This story is going to be MUCH better with some smoke and other visual effects. I can't wait until summer 2019!

★☆☆☆☆ Too advanced for me …

I bought this in the hope that I would be able to sit down on a rainy Sunday afternoon and look through it, and with each passing page, feel a sense of achievement.

However, for the life of me, I have been unable to find **Waldo** on any single page yet!

Someone, please give me some hints!

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00000    10097 32533    76520 13586    34673 54876    80959 09117    39292 74945
00001    37542 04805    64894 74296    24805 24037    20636 10402    00822 91665
00002    08422 68953    19645 09303    23209 02560    15953 34764    35080 33606
00003    99019 02529    09376 70715    38311 31165    88676 74397    04436 27659
00004    12807 99970    80157 36147    64032 36653    98951 16877    12171 76833

00005    66065 74717    34072 76850    36697 36170    65813 39885    11199 29170
00006    31060 10805    45571 82406    35303 42614    86799 07439    23403 09732
00007    85269 77602    02051 65692    68665 74818    73053 85247    18623 88579
00008    63573 32135    05325 47048    90553 57548    28468 28709    83491 25624
00009    73796 45753    03529 64778    35808 34282    60935 20344    35273 88435

00010    98520 17767    14905 68607    22109 40558    60970 93433    50500 73998
00011    11805 05431    39808 27732    50725 68248    29405 24201    52775 67851
00012    83452 99634    06288 98083    13746 70078    18475 40610    68711 77817
00013    88685 40200    86507 58401    36766 67951    90364 76493    29609 11062
00014    99594 67348    87517 64969    91826 08928    93785 61368    23478 34113

00015    65481 17674    17468 50950    58047 76974    73039 57186    40218 16544
00016    80124 35635    17727 08015    45318 22374    21115 78253    14385 53763
00017    74350 99817    77402 77214    43236 00210    45521 64237    96286 02655
00018    69916 26803    66252 29148    36936 87203    76621 13990    94400 56418
00019    09893 20505    14225 68514    46427 56788    96297 78822    54382 14598

00020    91499 14523    68479 27686    46162 83554    94750 89923    37089 20048
00021    80336 94598    26940 36858    70297 34135    53140 33340    42050 82341
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00023    12550 73742    11100 02040    12860 74697    96644 89439    28707 25815
00024    63606 49329    16505 34484    40219 52563    43651 77082    07207 31790
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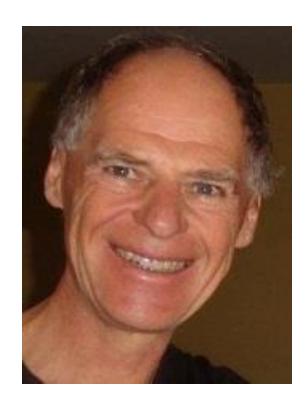
Generating Random Numbers

Anyone who attempts to generate random numbers by deterministic means is, of course, living in a state of sin.

John von Neumann

The numbers in **OMRD** were obtained electro-mechanically (pulse generator + punch machine).

Nowadays random numbers are generated on a computer via simple mathematical formulas.

Pierre L'Ecuyer

mrg32k3a

Makoto Matsumoto

Mersenne Twister

Basic Random Number Generator

1. Choose numbers *m*, *a* and *s(0)* (between *0* and *m*). Set *t = 1*.

2. Let *s(t)* be the **remainder** of $a \times s(t-1)$, when **divided** by *m*.

3. Output *u(t) = s(t)/m.*

4. Increase *t* by 1 and repeat from Point 2.

Example

m = 7, a = 3, s(0) = 1, produces:

s(1) = 3
s(2) = 2
s(3) = 6
s(4) = 4
s(4) = 5
s(5) = 1
s(6) = 3,…

Good choice:
m = 2^(31) − 1 = 2147483647
a = 7^5 = 16807
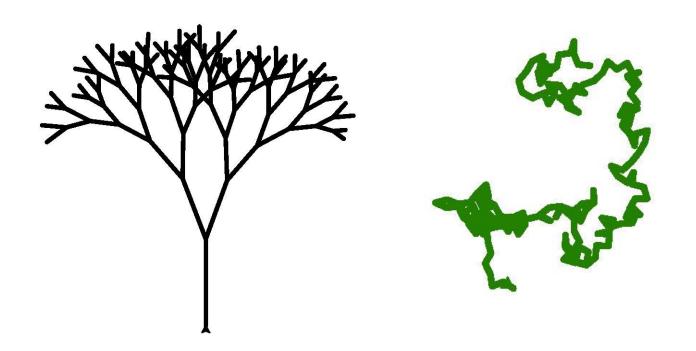
What can we do with Random Numbers?
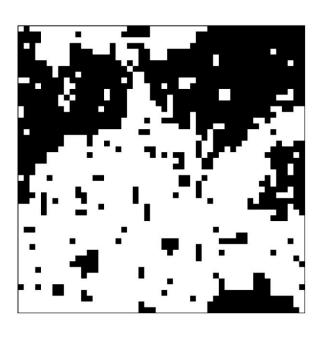
- **Simulate**: carry out random experiments

- **Estimate**: calculate numbers using random experiments

- **Optimise**: search using random processes

Some Random Process Simulations

Monte Carlo Simulation
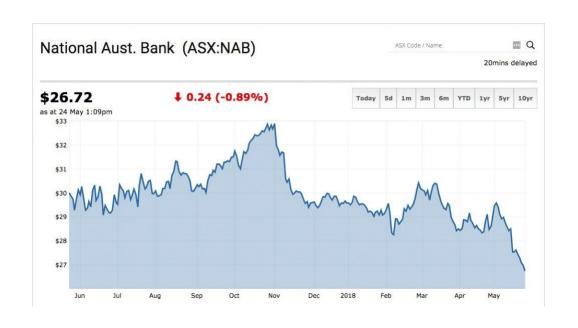
- How can we simulate 10 coin flips?
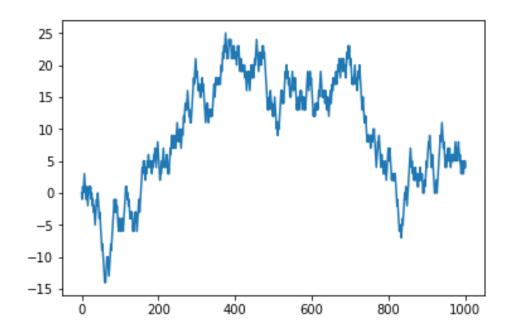
- Use the `rand` function (`firstrand.ipynb`).

- Simulate a 2-D random walk (`turtle.ipynb`)

- Simulate a tree (`tree.ipynb`)
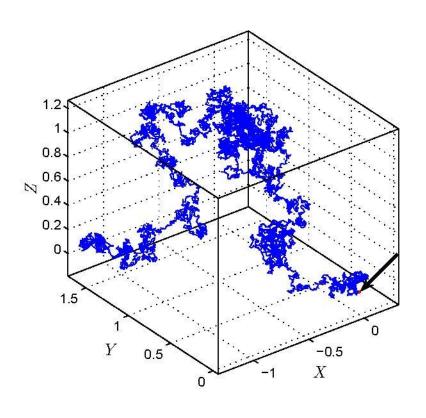
Share Price: a Random Walk?

randwalk.ipynb
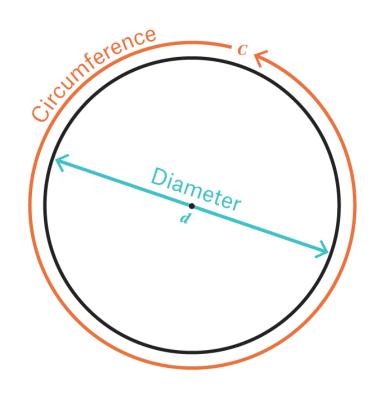
Simulating Brownian Motion
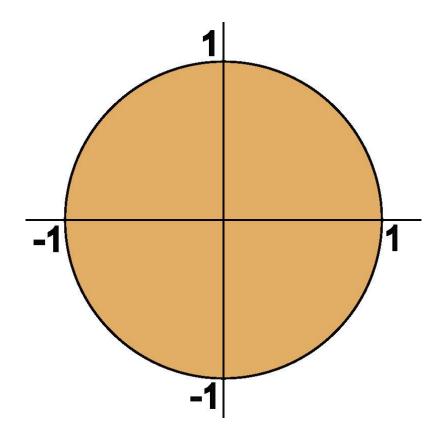
Monte Carlo for Estimation: Evaluating Pi

Two Definitions/Properties of π

$$\pi = C/D$$

$$\pi = \text{Area of Unit Disk}$$

Calculating π

- Archimedes (250 BC): $\pi \approx 3.14$

- Leibniz (1680) $\quad \dfrac{\pi}{4} = 1 \; - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

- Ramanujan: amazing formulas for π

- More than 22,000,000,000,000 (22 trillion) digits known.

Indiana Pi Bill (1897)

- Edwin Goodwin thought he had "squared the circle"
- As a by-product he deduced that Pi = 3.2 (exactly!)
- A Bill was introduced in House of Representatives to legislate this value for Pi. The bill was passed without a dissenting vote.
- In the Senate, it nearly passed. However, the bill was postponed indefinitely, with the help of:

   Clarence Abiathar Waldo
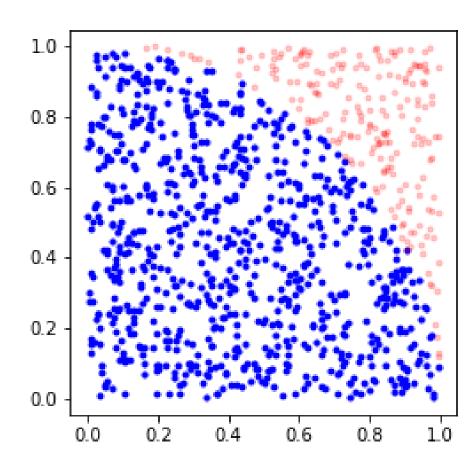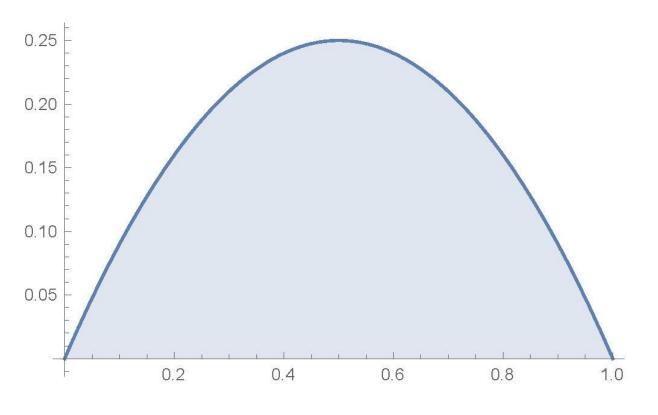
CLARENCE ABIATHAR WALDO,
A. M., Ph. D.,
Head Professor of Mathematics.

Estimating Pi with Monte Carlo

- **simulatepi.ipynb**

Evaluating Areas/Volumes/Integrals

Area under
x(1-x)

mcint.ipynb
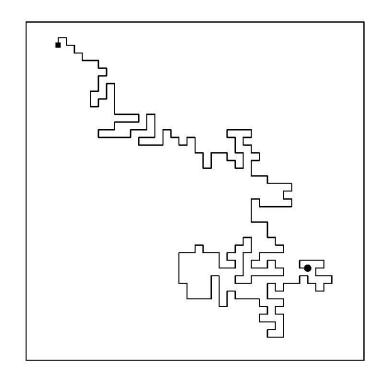
Monte Carlo for Searching and Counting
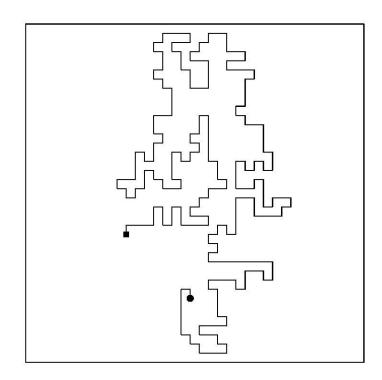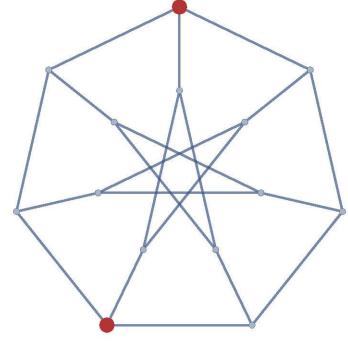
Counting Complicated Objects

How Many Paths?

Idea for Counting

- Suppose the "special" objects all lie in a larger set for which we know the number of elements exactly, say N.

- Randomly select n objects from the larger set.

- Count how many are special, say s.

- Then we estimate the number of special objects as $N\, s/n$

Idea for Counting

- The previous method is very inefficient when the number of special objects is very small compared to N.

- More sophisticated methods are then required.
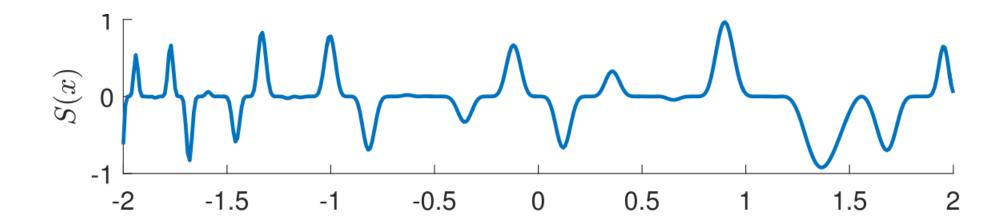
Monte Carlo for Optimisation
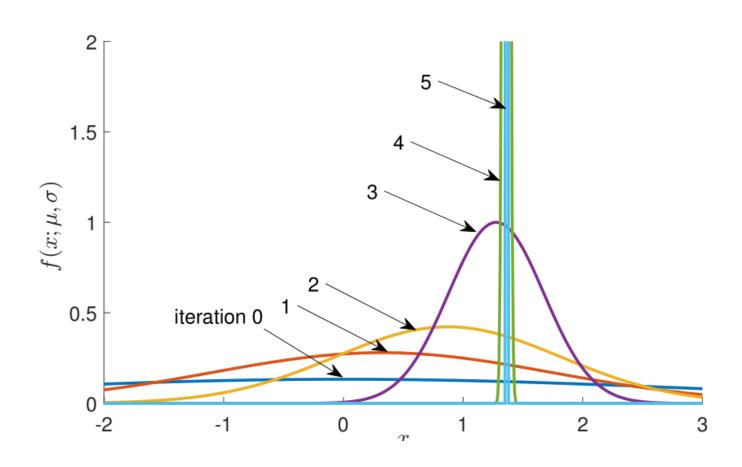
Idea for Optimisation

- Simulate many points on the search space according to a random mechanism
- Take the best performing points
- Update the random mechanism based on the best performing points
- Repeat until the search "degenerates" to one point

Minimize this function
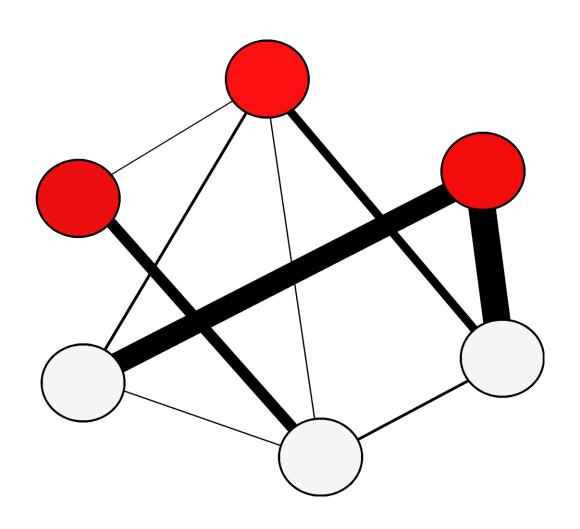
Evolution of the Cross-Entropy Algorithm

How to Split a Group

How to Split a Group Optimally
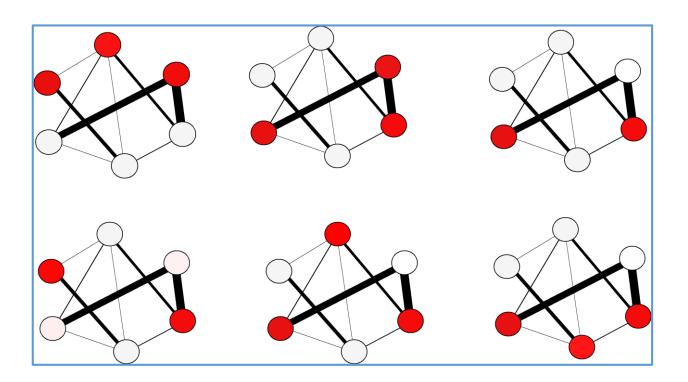
1. Randomly split the group in two, using biased coins
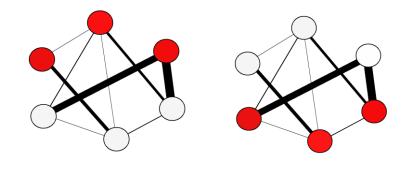
2. Do this many times

3. Select the few best

4. Update the coins, based on the best ones found.

5. Repeat
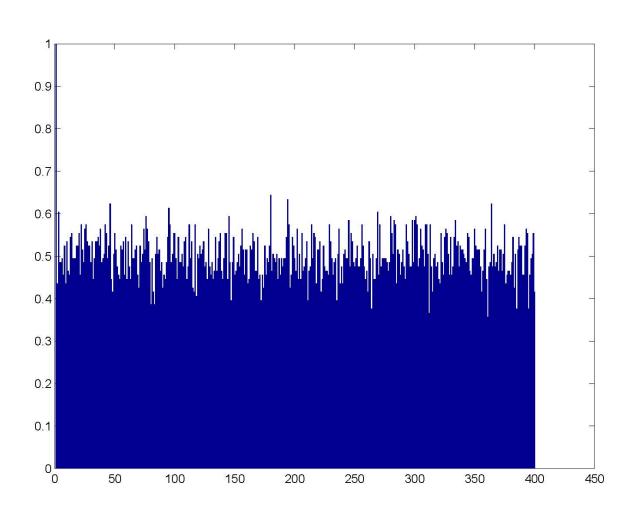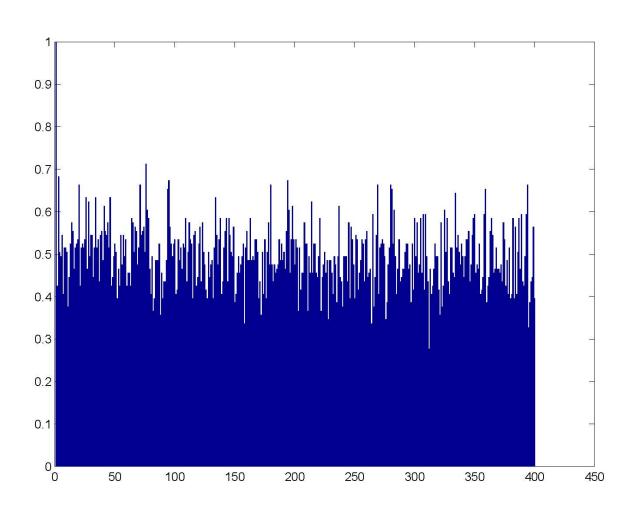
Example
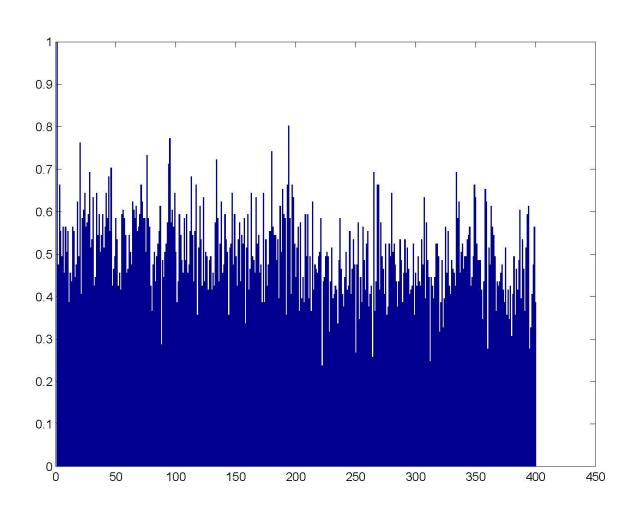
- Split a group of 400 people
- There are 2^400 possible groups; that is:
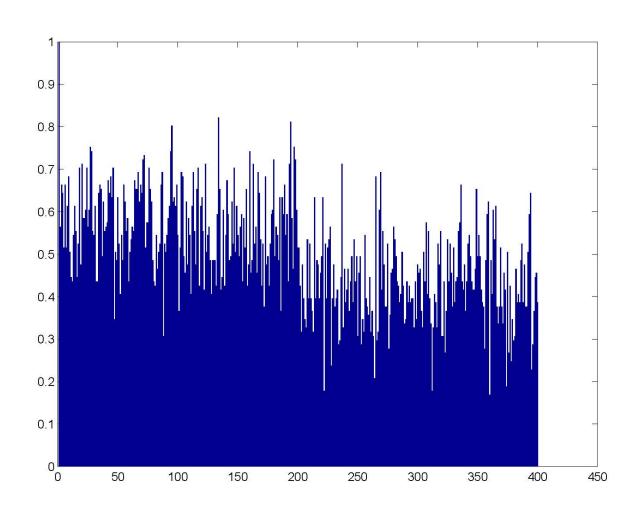
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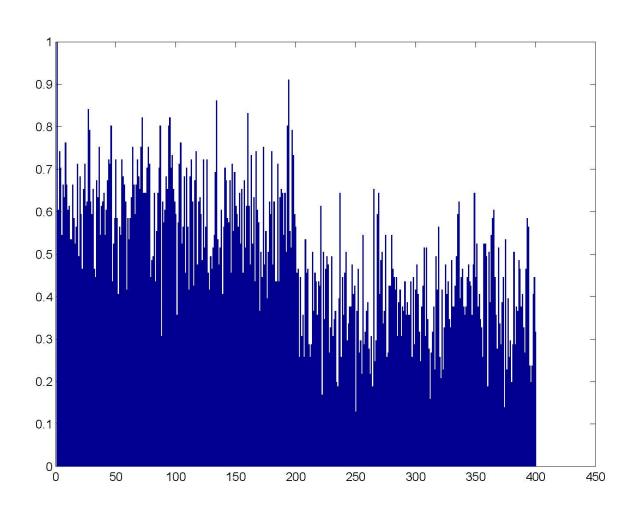
- Take 400 coins; start with Heads probabilities all 0.5
- Flip a coin for person 1,…,400 to put into Heads or Tails group.
- Do this 1000 times, take the best 100, update the coins

Evolution of the coin biases. Step 1

Evolution of the coin biases. Step 2

Evolution of the coin biases. Step 3

Evolution of the coin biases. Step 4

Evolution of the coin biases. Step 5

Evolution of the coin biases. Step 6

Evolution of the coin biases. Step 7

Evolution of the coin biases. Step 8

Evolution of the coin biases. Step 9

Evolution of the coin biases. Step 10

Evolution of the coin biases. Step 11

Evolution of the coin biases. Step 12

Evolution of the coin biases. Step 13

Evolution of the coin biases. Step 14

Evolution of the coin biases. Step 15

Evolution of the coin biases. Step 16

Evolution of the coin biases. Step 17

Evolution of the coin biases. Step 18

Evolution of the coin biases. Step 19
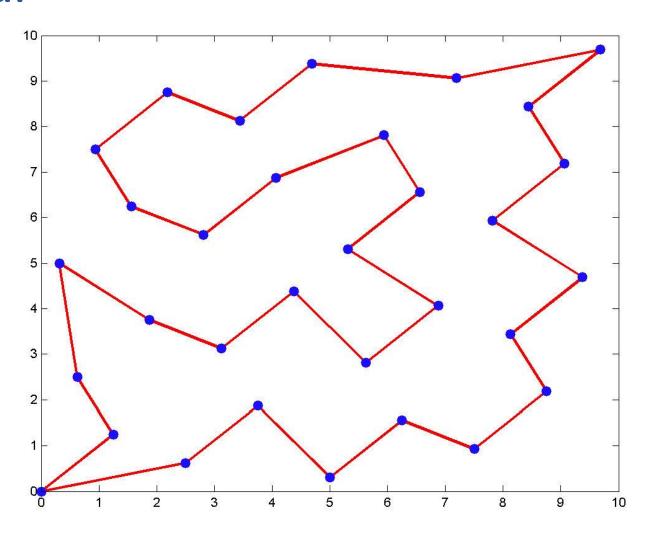
Evolution of the coin biases. Step 20

What is the best tour?

Similar idea for optimisation

- Generate paths/tours randomly via some random procedure

- Take the best (shortest) paths

- Update the random procedure using the best tour, to produce "better" paths next time
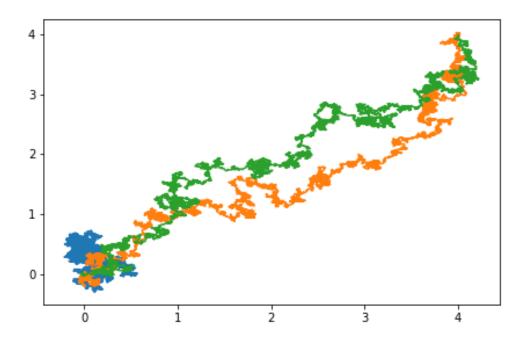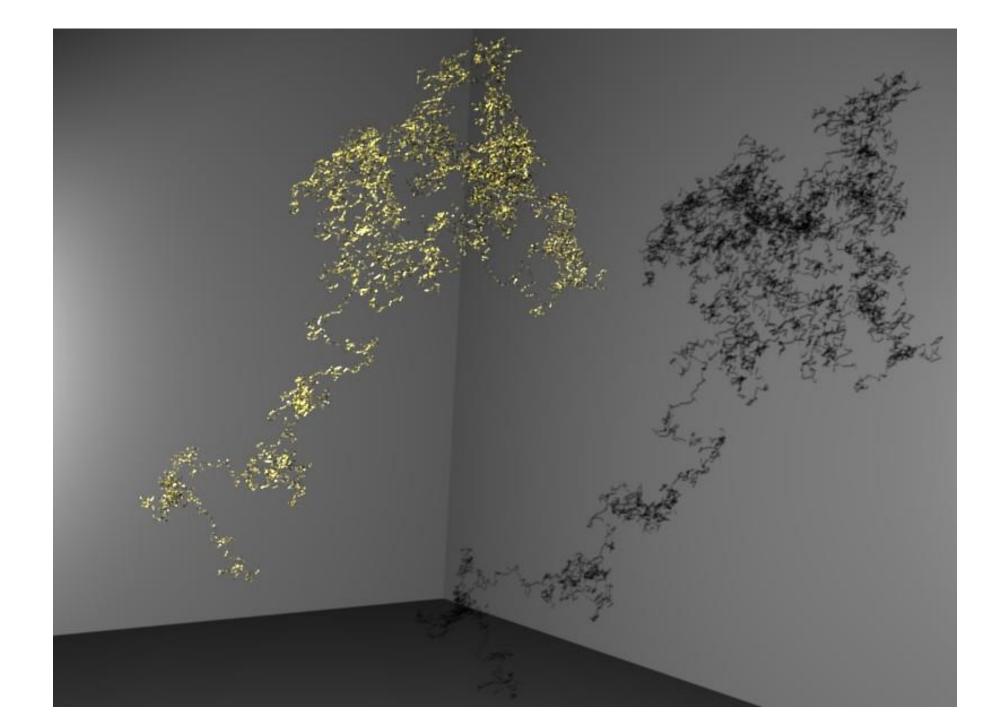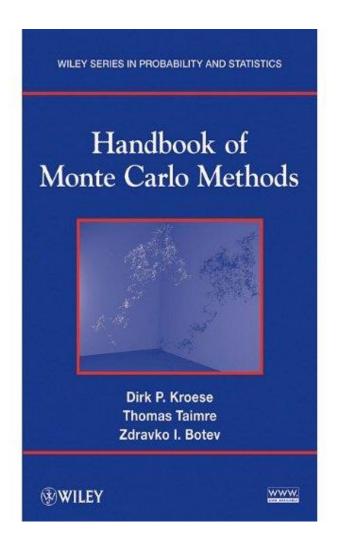
- Repeat until convergence
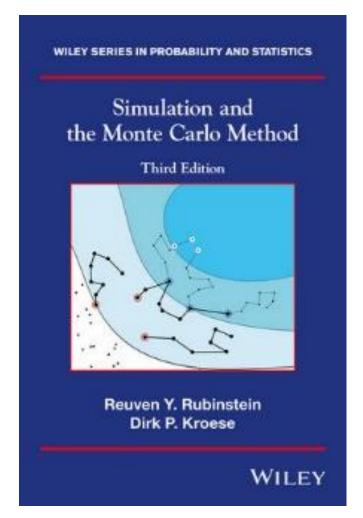
Best Tour

Finally …

- Monte Carlo guidelines:
  - Set off in a random direction sometimes …
  - Diversity is crucial!

- **Determinism** vs **Free Will**
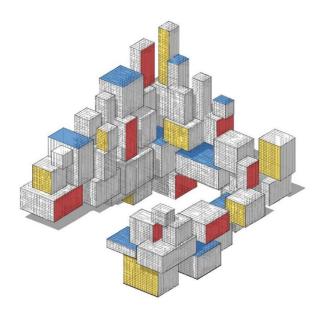 - Conditional random walk

Coming soon!

Mathematical and Statistical Methods for Data Science and Machine Learning

Dirk P. Kroese, Zdravko I. Botev, Thomas Taimre, and Radislav Vaisman